Second Order Linear Equations

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In first part of this chapter, we consider second order linear ordinary linear equations, i.e., a differential equation of the form

$$L[y] = \frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = g(t).$$

The above equation is said to be **homogeneous** if $g(t) = 0$ and the equation

$$L[y] = 0$$

is called the **associated homogeneous equation**.

Theorem (Existence and uniqueness of solution)

Let I be an open interval and $t_o \in I$. Let $p(t), q(t)$ and $g(t)$ be continuous functions on I. Then for any real numbers y_0 and y_0', the initial value problem

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), & t \in I \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases},$$

has a unique solution on I.

Theorem (Principle of superposition)

If y_1 and y_2 are two solutions of the homogeneous equation

$$L[y] = 0,$$

then $c_1y_1 + c_2y_2$ is also a solution for any constants c_1 and c_2.

The principle of superposition implies that the solutions of a homogeneous equation form a vector space. This suggests us finding a basis for the solution space.

Definition

Two functions $u(t)$ and $v(t)$ are said to be **linearly dependent** *if there exists constants k_1 and k_2, not both zero, such that $k_1 u(t) + k_2 v(t) = 0$ for all $t \in I$. They are said to be* **linearly independent** *if they are not linearly dependent.*

Definition (Fundamental set of solutions)

We say that two solutions y_1 and y_2 form a **fundamental set of solutions** *of the homogeneous equation $L[y] = 0$ if they are linearly independent.*

Definition (Wronskian)

Let y_1 and y_2 be two differentiable functions. Then we define the **Wronskian** *(or* **Wronskian determinant***) to be the function*

$$W(t) = W(y_1, y_2)(t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

Theorem

Let $u(t)$ and $v(t)$ be two differentiable functions on open interval I. If $W(u, v)(t_0) \neq 0$ for some $t_0 \in I$, then u and v are linearly independent.

Proof.

Suppose $k_1 u(t) + k_2 v(t) = 0$ for all $t \in I$ where k_1, k_2 are constants. Then we have

$$\begin{cases} k_1 u(t_0) + k_2 v(t_0) & = & 0, \\ k_1 u'(t_0) + k_2 v'(t_0) & = & 0. \end{cases}$$

In other words,

$$\begin{pmatrix} u(t_0) & v(t_0) \\ u'(t_0) & v'(t_0 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now the matrix

$$\begin{pmatrix} u(t_0) & v(t_0) \\ u'(t_0) & v'(t_0) \end{pmatrix}$$

is non-singular since its determinant $W(u, v)(t_0)$ is non-zero by the assumption. This implies that $k_1 = k_2 = 0$. Therefore $u(t)$ and $v(t)$ are linearly independent. □

Remark: The converse is false, e.g. $u(t) = t^3$, $v(t) = |t|^3$.

Example

$y_1(t) = e^t$ *and* $y_2(t) = e^{-2t}$ *form a fundamental set of solutions of*

$$y'' + y' - 2y = 0$$

since $W(y_1, y_2) = e^t(-2e^{-2t}) - e^t(e^{-2t}) = -3e^{-t}$ *is not identically zero.*

Example

$y_1(t) = e^t$ *and* $y_2(t) = te^t$ *form a fundamental set of solutions of*

$$y'' - 2y' + y = 0$$

since $W(y_1, y_2) = e^t(te^t + e^t) - e^t(te^t) = e^{2t}$ *is not identically zero.*

Example

The functions $y_1(t) = 3$, $y_2(t) = \cos^2 t$ *and* $y_3(t) = -2\sin^2 t$ *are linearly dependent since*

$$2(3) + (-6)\cos^2 t + 3(-2\sin^2 t) = 0.$$

One may justify that the Wronskian

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = 0.$$

Example

Show that $y_1(t) = t^{1/2}$ *and* $y_2(t) = t^{-1}$ *form a fundamental set of solutions of*

$$2t^2y'' + 3ty' - y = 0, \quad t > 0.$$

Solution: It is easy to check that y_1 and y_2 are solutions to the equation. Now

$$W(y_1, y_2)(t) = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -\frac{3}{2}t^{-3/2}$$

is not identically zero. We conclude that y_1 and y_2 form a fundamental set of solutions of the equation. □

Theorem (Abel's Theorem)

If y_1 and y_2 are solutions of the equation

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval I, then

$$W(y_1, y_2)(t) = c \exp\left(-\int p(t)dt \right),$$

where c is a constant that depends on y_1 and y_2. Further, $W(y_1, y_2)(t)$ is either identically zero on I or never zero on I.

Proof.

Since y_1 and y_2 are solutions, we have

$$\begin{cases} y_1'' + p(t)y_1' + q(t)y_1 &= 0 \\ y_2'' + p(t)y_2' + q(t)y_2 &= 0. \end{cases}$$

If we multiply the first equation by $-y_2$, multiply the second equation by y_1 and add the resulting equations, we get

$$\begin{aligned} (y_1 y_2'' - y_1'' y_2) + p(t)(y_1 y_2' - y_1' y_2) &= 0 \\ W' + p(t)W &= 0 \end{aligned}$$

which is a first-order linear and separable differential equation with solution

$$W(t) = c \exp\left(-\int p(t)dt\right),$$

where c is a constant. Since the value of the exponential function is never zero, $W(y_1, y_2)(t)$ is either identically zero on I (when $c = 0$) or never zero on I (when $c \neq 0$). □

Theorem

Suppose y_1 and y_2 are solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous on an open interval I. Then y_1 and y_2 are linearly independent if and only if $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$.

Proof.

The "if" part follows by Theorem 1.6. To prove the "only if" part, suppose $W(y_1, y_2)(t) = 0$ for any $t \in I$. Take any $t_0 \in I$, we have

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = 0.$$

Then system of equations

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) & = & 0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) & = & 0 \end{cases},$$

has non-trivial solution for c_1, c_2. Now the function $c_1 y_1 + c_2 y_2$ is a solution to the initial value problem

$$\begin{cases} y'' + p(t)y' + q(t)y = 0, \quad t \in I, \\ y(t_0) = 0, \quad y'(t_0) = 0. \end{cases}$$

This initial value problem has a solution $y(t) \equiv 0$ which is unique by Theorem 1.1. Thus $c_1 y_1 + c_2 y_2$ is identically zero and therefore y_1, y_2 are linearly dependent. □

Theorem

Let y_1 and y_2 be solutions of

$$L[y] = y'' + p(t)y' + q(t)y = 0, \;\; t \in I$$

where p and q are continuous on an open interval I. Then $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$ if and only if every solution of the equation is of the form $c_1 y_1 + c_2 y_2$ for some constants c_1, c_2.

proof

Suppose $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$. Let $y = y(t)$ be a solution of of $L[y] = 0$ and write $y_0 = y(t_0)$, $y_0' = y'(t_0)$. Since $W(t_0) \neq 0$, there exists constants c_1, c_2 such that

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}.$$

Now both y and $c_1 y_1 + c_2 y_2$ are solution to the initial problem

$$\begin{cases} y'' + p(t)y' + q(t)y = 0, & t \in I, \\ y(t_0) = y_0, & y'(t_0) = y_0'. \end{cases}$$

Therefore $y = c_1 y_1 + c_2 y_2$ by the uniqueness part of Theorem 1.1.

Proof

Suppose the general solution of $L[y] = 0$ is $y = c_1 y_1 + c_2 y_2$. Take any $t_0 \in I$. Let u_1 and u_2 be solutions of $L[y] = 0$ with initial values

$$\begin{cases} u_1(t_0) &= 1 \\ u_1'(t_0) &= 0 \end{cases} \text{ and } \begin{cases} u_2(t_0) &= 0 \\ u_2'(t_0) &= 1 \end{cases}.$$

The existence of u_1 and u_2 is guaranteed by Theorem 1.1. Thus exists constants $a_{11}, a_{12}, a_{21}, a_{22}$ such that

$$\begin{cases} u_1 &= a_{11} y_1 + a_{21} y_2 \\ u_2 &= a_{12} y_1 + a_{22} y_2 \end{cases}.$$

In particular, we have

$$\begin{cases} 1 & = & u_1(t_0) & = & a_{11}y_1(t_0) + a_{21}y_2(t_0) \\ 0 & = & u_2(t_0) & = & a_{12}y_1(t_0) + a_{22}y_2(t_0) \end{cases}$$

and

$$\begin{cases} 0 & = & u_1'(t_0) & = & a_{11}y_1'(t_0) + a_{21}y_2'(t_0) \\ 1 & = & u_2'(t_0) & = & a_{12}y_1'(t_0) + a_{22}y_2'(t_0) \end{cases}.$$

In other words,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Therefore the matrix

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}$$

is non-singular and its determinant $W(y_1, y_2)(t_0)$ is non-zero. □

Theorem

Let $L[y] = y'' + p(t)y' + q(t)y$, where $p(t)$ and $q(t)$ are continuous on an open interval I. The solution space of the homogeneous equation $L[y] = 0,\ t \in I$ is of dimension two. Let y_1 and y_2 be two solutions of $L[y] = 0$, then the following statements are equivalent.

1. $W(y_1, y_2)(t_0) \neq 0$ *for some* $t_0 \in I$.
2. $W(y_1, y_2)(t) \neq 0$ *for all* $t \in I$.
3. *The functions y_1 and y_2 form a fundamental set of solutions, i.e., y_1 and y_2 are linearly independent.*
4. *Every solution of the equation is of the form $c_1y_1 + c_2y_2$ for some constants c_1, c_2, i.e., y_1 and y_2 span the solution space of $L[y] = 0$.*
5. *The functions y_1 and y_2 constitute a basis for the solution space of $L[y] = 0$.*

Proof

The only thing we need to prove is that there exists solutions with $W(t_0) \neq 0$ for some $t_0 \in I$. Take any $t_0 \in I$. By Theorem 1.1, there exists solutions y_1 and y_2 to the homogeneous equation $L[y] = 0$ with initial conditions $y_1(t_0) = 1,\ y_1'(t_0) = 0$ and $y_2(t_0) = 0,\ y_2'(t_0) = 1$ respectively. Then $W(y_1, y_2)(t_0) = \det(I) = 1 \neq 0$ and we are done. □